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The structure of a weak shock wave undergoing reflexion from a wall

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The Navier–Stokes equations are used to study the unsteady structure of a weak shock wave reflecting from a plane wall. Both an adiabatic and an isothermal wall are considered. Incident and reflected shock structures are found by expanding the dependent variables in asymptotic series in the shock strength; the first-order terms are shown to satisfy an equation analogous to Burgers equation. The structure of the wave during reflexion is obtained from an expansion in which the first-order terms satisfy the acoustic equations. The isothermal wall boundary condition requires the introduction of a thermal layer adjacent to the wall. In this case viscosity and convection play a role secondary to the wall temperature boundary condition in determining the structure of the reflected wave. The presentation is simplified by introducing a generalized Burgers equation that gives the same first-order results as the Navier–Stokes equations. Correct second-order results are obtained from this equation simply by applying a correction to the result for the temperature.

1. Introduction

Shock-wave phenomena have intrigued the physicist and applied mathematician for almost a century. This enduring interest has been motivated by the intrinsic non-linearity and thermodynamic irreversibility of the shock process. At first investigators treated shock waves as discontinuities governed by the conservation laws of mass, momentum and energy. Our understanding of shock-wave phenomena was advanced by Taylor (1910), who examined the viscous structure of weak shock waves using the Navier–Stokes equations. Taylor's work showed that the irreversible process of viscous diffusion balances the non-linear process of convection to maintain the structure of a shock wave. His work was concerned solely with steady-state shock waves. In the last few decades, work on the viscous structure of weak shock waves has been extended to the unsteady case by Lagerstrom, Cole & Trilling (1949), Hayes (1956), Lighthill (1956), Moran & Shen (1966) and others. The work was greatly aided by the development of an equation which took account of both convective and diffusive effects. This equation was first proposed by Burgers (1948) as a simple model of the Navier–Stokes equation and was used by him to study the interplay of convective and diffusive effects in a turbulent fluid. Later it was shown that Burgers'

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equation provided a good approximation for the study of finite amplitude waves with viscosity, and the equation was used to examine a whole family of problems ranging from shock formation to shock coalescence. Lighthill's article provides a survey of this work.

In this paper we concern ourselves with the problem of the structure of a shock wave as the wave reflects from a wall. Like the interaction of two shock waves of different families, i.e. with intersecting paths, and unlike the coalescence of two shock waves of the same family, the reflexion of a shock wave from a wall must be described by an equation, which simultaneously takes account of wave motion in two directions. Burgers equation contains a first-order wave operator; hence this equation is able to handle only one direction of wave motion. Also, the approximations that lead to Burgers equation imply an adiabatic flow; consequently only an insulated boundary may be treated directly with this equation. We overcome these difficulties by matching asymptotic expansions of the Navier–Stokes equations: the expansions are carried out in a parameter that is a measure of the shock strength; different expansions are required for various domains of space and time. When the shock wave is far from the wall, the wall being to the right of the shock wave, the first term in an asymptotic expansion of the solution is governed either by a right-going Burgers equation for the incident shock, or by a left-going Burgers equation for the reflected shock. During the reflexion period the first term in the expansion is controlled by the equations of acoustics. A thermal region arises when we treat the problem of reflexion from an isothermal wall. The formalism developed here clearly indicates how higher-order terms in the expansion may be found, and points the way to the solution of a number of similar problems. Furthermore, there are indications that other classes of non-linear wave problems will yield to the methods presented below.

The structure of a reflecting shock wave provides a problem which requires an extension of the techniques available for the study of weak shock waves. It is also of fundamental interest to the fluid mechanist. Recent work by Baganoff (1964, 1965) describes the measurement of the pressure history of a shock wave reflecting off the end-wall of a shock tube. Petty (1966) and Scala & Gordon (1966) have carried out numerical solutions of the reflecting shock structure. Goldsworthy (1959) studied the effect of a conducting wall on the trajectory of a shock wave of arbitrary strength. Goldsworthy's theory has recently been extended and improved by Clarke (1967). Spence (1961) has suggested the use of methods employing Burgers equation in carrying out chemical rate calculations during the shock reflexion period.

We begin by formulating the reflected shock problem and by outlining the method of solution. Instead of proceeding to solve the problem directly from the Navier–Stokes equations, which we could do, we introduce a simpler equation, both for its own intrinsic interest and to simplify the presentation. In the appendix we show that *the results are the same as those that would be obtained from the complete equations.* We call this simpler equation the *generalized Burgers equation* because it is accurate to the same order as Burgers equation in regions where Burgers equation is applicable, and because it generalizes Burgers equation to encompass wave motion in both directions. First we treat the reflexion of a weak shock off an adiabatic wall. This is followed by a treatment of the isothermal wall problem. Finally, the last section contains a discussion of our results and some suggestions for future work.

2. Formulation

We assume the shock wave to be fully formed long before reflexion is to occur so that its initial structure, when viewed in shock-fixed co-ordinates, can be computed from the steady form of the Navier–Stokes equations. Such a steady solution provides physically meaningful initial conditions for the wall reflexion problem, but does not satisfy exactly the wall boundary condition of zero velocity. This is not an unreasonable situation, because initially the shock can be located far enough away from the wall so that the gas velocity at the wall is exponentially small in the distance co-ordinate. However, it is both mathematically and conceptually simpler to impose the asymptotic condition that, as the time until reflexion becomes infinitely large, the shock structure should approach the classical steady-state structure. Long after reflexion the structure should once more approach a steady state, and this will provide a check on our solution. To formulate the initial conditions precisely, reflexion is considered to occur at time $t^* = 0$, when, if $u^*(x^*, t^*)$ is the gas velocity, the spatial gradient $u^*_{x^*}$ is a maximum. The wall location is taken to be at $x^* = 0$ and the shock is assumed to approach the wall from $x^* = -\infty$ (see figure 1). In terms of these co-ordinates, as time goes to minus infinity the solution must approach the steady-state shock structure required by the given strength of the incident shock and the initial thermodynamic state of the quiescent gas between the wall and the shock. We consider the shock strength as measured by the parameter $\epsilon = u^*_1/a^*_0$ to be small, where a^*_0 is the sound speed ahead of the incident shock, and u^*_1 is the velocity behind the shock. The subscripts 0 and 1 are used to denote conditions ahead of and behind the incident shock.

The zero velocity condition at the wall is not a sufficient condition for a unique solution to the problem; we must also specify the heat transfer properties of the wall. The cases of theoretical interest here are the insulated or adiabatic wall and the isothermal wall. The adiabatic solution must be known to solve the isothermal wall problem; consequently this case is treated first. An adiabatic wall implies no heat transfer; thus this problem is equivalent to the collision of two equal strength shock waves. The isothermal boundary condition provides a reasonable model of shock tube end-wall reflexion because, for the times of interest, heat transfer to the wall is small and the wall acts as a constant temperature heat sink.

We now seek a solution to the problem of a shock wave reflecting off an adiabatic wall by constructing an asymptotic expansion for small shock strengths, i.e. small ϵ. The theory of weak shock waves tells us that the shock thickness is $O(\epsilon^{-1})$. This result, coupled with the equations of motion, shows that the characteristic times for the diffusion and for the steepening of a wave are $O(\epsilon^{-2})$. On the other hand, the characteristic time for the duration of the shock-wall interaction is the thickness of the shock divided by its velocity. Therefore the interaction occurs in a time $O(\epsilon^{-1})$ of the characteristic time for diffusion and convection, and it is reasonable to anticipate that the phenomenon of shock reflexion is encompassed by the equations of linear acoustics. In order to account formally for this acoustic behaviour, the Navier–Stokes equations are written in terms of variables for which the limit process $\epsilon \to 0$ reduces the Navier–Stokes equations to the equation of acoustics. The appropriate non-dimensional independent variables are based on the shock thickness and on the time for a sound pulse to traverse this distance:

$$x = x^* \rho_0^* a_0^* \epsilon / \mu_0^*, \quad t = t^* \rho_0^* a_0^{*2} \epsilon / \mu_0^*.$$

Here x and t are the acoustic space and time co-ordinates, μ_0^* the longitudinal coefficient of viscosity of the quiescent fluid, and ρ_0^* and a_0^* are the quiescent density and sound speed. The gas is considered ideal and calorically perfect so that $a_0^{*2} = \gamma R T_0^*$ where γ, the ratio of specific heats is assumed constant, R is the gas constant, and T_0^* is the temperature. The acoustic velocity, pressure, density, temperature and viscosity are defined by

$$u = u^*/\epsilon a_0^*, \quad p^* = p_0^*(1+\epsilon p),$$
$$\rho^* = \rho_0^*(1+\epsilon\rho), \quad T^* = T_0^*(1+\epsilon T),$$

and

$$\mu^* = \mu_0^*(1+\epsilon\mu).$$

The Navier–Stokes equations written in acoustic co-ordinates are

$$\rho_t + u_x + \epsilon(\rho u)_x = 0, \tag{2.1a}$$

$$u_t + p_x/\gamma + \epsilon(-u_{xx} + \rho u_t + u u_x) + \epsilon^2(\rho u u_x - (\mu u_x)_x) = 0, \tag{2.1b}$$

$$T_t + (\gamma-1)u_x + \epsilon(-\gamma T_{xx}/Pr + \rho T_t + u T_x + (\gamma-1) p u_x)$$
$$+ \epsilon^2[-\gamma(\gamma-1)(4/3+\delta)u_x^2 - \gamma(\mu T_x)_x/Pr + \rho u T_x]$$
$$- \epsilon^3 \gamma(\gamma-1)(4/3+\delta)\mu u_x^2 = 0, \tag{2.1c}$$

and

$$p = \rho + T + \epsilon \rho T. \tag{2.1d}$$

The Prandtl number, Pr, and the specific heat at constant pressure are assumed to be constant; δ is the ratio of the bulk viscosity to the usual viscosity. Note that if ϵ goes to zero with the acoustic co-ordinates fixed, the Navier–Stokes equations reduce to the equations of acoustics.

For an adiabatic wall the boundary conditions are

$$u(0,t;\epsilon) = 0 \quad \text{and} \quad T_x(0,t;\epsilon) = 0.$$

In the case of an isothermal wall the last condition is replaced by $T(0,t;\epsilon) = 0$. The initial conditions are imposed by requiring the solution to approach a given steady-state shock structure as $t \to -\infty$. We begin with an attempt to obtain an asymptotic expansion for the structure of the reflecting shock wave by expanding the dependent variables, when expressed in acoustic co-ordinates, in a series in ϵ. For example,

$$u(x,t;\epsilon) = \sum_{n=1}^{\infty} u^{(n)}(x,t)\epsilon^{n-1}.$$

Such a scheme cannot be uniformly valid; we employ it here only to provide the motivation for the correct procedure. Instead of substituting the above series into the Navier–Stokes equation and equating like powers of ϵ to obtain a hierarchy of equations for the terms in the series, we define an acoustic limiting process in which $\epsilon \to 0$ with the acoustic variables fixed. This limiting process, which we will denote by 'lim', permits us to obtain the terms in the acoustic expansion by operations such as

$$u^{(1)}(x,t) = \lim u(x,t;\epsilon) = \operatorname*{limit}_{\epsilon \to 0} u(x,t;\epsilon)$$

and

$$u^{(2)}(x,t) = \lim \left(\frac{u(x,t;\epsilon) - u^{(1)}(x,t)}{\epsilon} \right).$$

We also find the partial expansion notation

$$E_m u(x,t;\epsilon) = \sum_{n=1}^{m} u^{(n)}(x,t)\, \epsilon^{n-1}$$

useful. The symbol E_m means: express the function u in acoustic co-ordinates, and then expand to m terms as a series in ϵ.

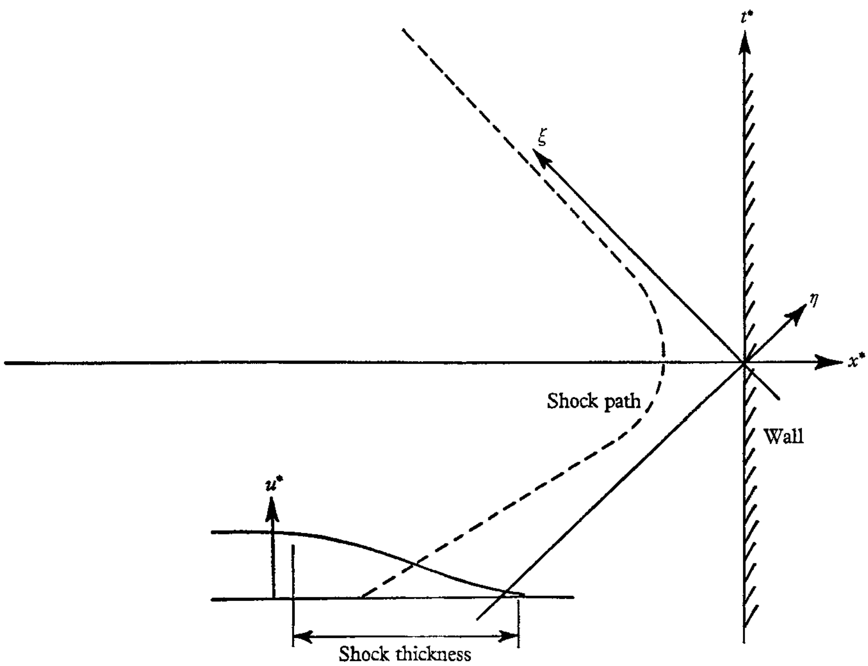

FIGURE 1. Sketch of co-ordinate systems, initial shock-wave structure and hypothetical shock path.

When the acoustic limit is applied to equations (2.1), we see that the first-order terms in the acoustic expansion are solutions of the ordinary acoustic equation, e.g.

$$u_{tt}^{(1)} - u_{xx}^{(1)} = 0.$$

The initial conditions for these first-order equations are obtained by applying the operation lim to the original initial conditions. The initial condition for u is that as $x, t \to -\infty$, we require $u(x,t;\epsilon)$ to approach the classical Taylor structure for weak shocks:

$$u(x, t \to -\infty) \to (1 + e^{(\Gamma/\beta)(x - t - \epsilon\Gamma t/2)})^{-1}, \tag{2.2}$$

where $\Gamma = (\gamma+1)/2$ and $\beta = 1 + (\gamma-1)/Pr$. As $\epsilon \to 0$, we see that

$$u^{(1)} \to \lim u = f(x-t) \equiv (1 + e^{(\Gamma/\beta)(x-t)})^{-1}.$$

The letter f will be used throughout this paper to denote the function defined above. The solution of the wave equation which satisfies the above limiting condition and $u^{(1)}(0,t) = 0$ is

$$u^{(1)} = f(x-t) - f(-x-t).$$

If we attempt to calculate the second-order terms in the acoustic expansion, we find that $u^{(2)}(x,t)$ satisfies an inhomogeneous wave equation of the form

$$u^{(2)}_{tt} - u^{(2)}_{xx} = G(u^{(1)}, \rho^{(1)}, T^{(1)}).$$

The solution of the initial value problem for $u^{(2)}$ contains terms such as $f'(x-t)t$ and $f'(-x-t)t$, where the prime denotes differentiation with respect to the function's argument. These terms result from the inhomogeneous terms in the above wave equation. As $|t| \to \infty$ such terms are unbounded; thus when $t = O(1/\epsilon)$, the acoustic expansion for u becomes invalid because the second term in the expansion will be the same order in ϵ as the first term. Higher-order terms in the acoustic expansion exhibit even more rapid growth with time. The non-uniform behaviour of the acoustic expansion suggests that we need to define new variables which lead to an expansion that is valid as $|t|$ becomes infinite.

Consider the behaviour of the second-order acoustic solution as $t \to -\infty$, i.e. long before reflexion. When the right-facing characteristic $\xi = x - t$ is fixed and $-t$ becomes large, the term $tf'(-x-t)$ vanishes but the term $tf'(x-t)$ grows without bound. This motivates an expansion in which the co-ordinates ξ and $\bar{t} = \epsilon t$ are held fixed as $\epsilon \to 0$. This limit process will be denoted by $\overline{\lim}$, the symbol $\bar{E}_m u$ is used to designate the m-term expansion in ϵ of the function to its right in the variables $\xi, \bar{t}$. The co-ordinates ξ and $\bar{t}$ will be called the right-going Burgers co-ordinates because the equation satisfied by $\overline{\lim}\, u = \bar{u}^{(1)}(\xi, \bar{t})$ may be shown to be Burgers equation. To construct the appropriate expansion as $t \to \infty$, i.e. long after reflexion, we need to introduce left-going Burgers co-ordinates:† $\eta = (x+t)$ and $\bar{t} = \epsilon t$. These latter variables apply when $t > 0$, and the notations $\bar{E}_m u$ and $\overline{\lim}$ are also used for expansions of functions of these variables. This notation is a consistent one provided we adopt the rule that the co-ordinate ξ is used when $t < 0$, and η when $t > 0$.

Before we proceed with the formal construction of a uniformly valid asymptotic expression, let us briefly examine the manner in which the first term in the right-going Burgers variable expansion for u matches the first term in the acoustic expansion. If the Navier–Stokes equations are written in the variables ξ and $\bar{t}$, and, with these variables fixed, ϵ is made to approach zero, we see that $\bar{u}^{(1)}$ satisfies Burgers equation

$$\bar{u}^{(1)}_{\bar{t}} + \Gamma \bar{u}^{(1)} \bar{u}^{(1)}_{\xi} = \tfrac{1}{2}\beta \bar{u}^{(1)}_{\xi\xi}. \tag{2.3}$$

The derivation of this equation from (2.1), which is not straightforward, is discussed in §4. The steady-state solution of this equation with $\bar{u}^{(1)} \to 1$ as $\xi \to -\infty$ and $\bar{u}^{(1)} \to 0$ as $\xi \to +\infty$ is the Taylor shock structure $\bar{u}^{(1)} = f(\xi - \Gamma\bar{t}/2)$. When $\bar{t}$ goes to zero, this becomes the initial condition used for the first-order acoustic problem. Thus, for $t \to 0$ the first-order Burgers term approaches the first-order

† It may be easier for the reader to think in terms of 'incoming and outgoing' co-ordinates rather than in terms of our 'right-going and left-going' co-ordinates.

acoustic term for $t \to -\infty$, i.e. they match one another. A more formal way of carrying out this matching is to write the acoustic solution in terms of the Burgers co-ordinates and $\bar{u}^{(1)}(\xi, \bar{t})$ in terms of acoustic co-ordinates, so that

$$u^{(1)} = f(x-t) - f(-x-t) = f(\xi) - f(-\xi - 2\bar{t}/\epsilon),$$

and

$$\bar{u}^{(1)} = f(\xi - \Gamma\bar{t}/2) = f(x - t - \epsilon\Gamma t/2).$$

As ϵ goes to zero with $t < 0$, these two expressions become equal, a fact formally expressed by

$$\overline{\lim}\, u^{(1)}(x,t) = \lim \bar{u}^{(1)}(\xi, \bar{t}).$$

We will use a generalization of this 'limit matching principle',

$$E_m \bar{E}_n u = \bar{E}_n E_m u,$$

which Van Dyke (1964) calls the 'asymptotic matching principle'.

We now take the point of view that long before the shock nears the wall the problem should be viewed in the variables ξ, $\bar{t}$. The first-order solution then satisfies Burgers equation for right-going waves. This equation is not valid near the wall at any time because it does not permit solutions which take into account reflexions off the wall. Also, as the shock nears the wall the right-going Burgers equation becomes invalid for all space. Soon after reflexion, a left-going Burgers equation becomes applicable in a region away from the wall. These right- and left-going 'Burgers regions' are tied together by a region which is governed by the acoustic equations. The initial conditions in the acoustic region and in the left-going Burgers region are provided by the asymptotic matching principle: we assume that there exists an overlap domain between adjacent regions where the appropriate asymptotic expansions are both valid, and we demand that these solutions match one another.

3. A generalized Burgers equation

We could now proceed, using the Navier–Stokes equations, to obtain an asymptotic expansion in ϵ for the structure of a reflecting shock wave. However, both to simplify the presentation of our method and because of its own theoretical merit, we will introduce a simpler equation. This equation is as accurate as Burgers equation in the Burgers regions and takes into account wave motion in both directions. In the appendix the generalized Burgers equation is carefully compared with the Navier–Stokes equations. From this comparison it is clear that the first-order terms in an acoustic expansion of the solution to the generalized Burgers equation agree with the first-order terms in an acoustic expansion of the Navier–Stokes solution. The second-order results for the velocity and density also agree, but it is necessary to apply a correction to the second-order temperature and pressure results in order to bring them into agreement with the Navier–Stokes result. This correction is derived in the appendix. Thus the generalized Burgers equation, unlike Burgers equation, provides solutions which are uniformly valid in the entire (x,t)-plane. This generalized equation is also easily solved by numerical procedures, and later we will compare some numerical solutions with our analytical results. It is hoped that the equation will prove useful in the study of other finite wave problems involving wave motion in both directions.

Lighthill (1956) has shown that the equations of isentropic gas dynamics, with a linear diffusion term added to the momentum equation to account for viscosity and heat conduction, provide a description of finite wave motions in viscous fluids accurate to first order in the wave amplitude. These equations provide a starting-point for Lighthill's discussion of unsteady weak shock waves. By focusing his attention on problems involving weak waves moving in one flow direction, e.g. shock formation by a piston, Lighthill is able to simplify these equations. This is done by a process equivalent to the derivation of Burgers equation mentioned above, where the use of the Burgers limiting process resulted in Burgers equation. Lighthill's method is to write his equations in terms of the Riemann invariants of the inviscid equations. As is well known, one of these invariants is constant across a so-called simple wave region, that is, a region where disturbances are propagating along only one family of the isentropic characteristics. By showing that this fact remains true to first order in the wave amplitude even when the flow is viscous, Lighthill was able to reduce his equations to Burgers equation. Whether Burgers equation so derived is applicable to right-running or left-running waves depends on which of the two isentropic Riemann invariants is held constant. With the left-running invariant held constant, Burgers equation for right-running waves is obtained; with the right-running invariant held constant, Burgers equation for left-running waves results.

We now derive an equation which is simpler than the isentropic equations with a diffusion term, but which has embedded in its structure Burgers equation for either right- or left-running waves. First we rewrite Lighthill's equations in a form that suggests the introduction of a function which automatically satisfies the momentum equation. Such a function was introduced by von Mises (1958) for the isentropic inviscid case. The isentropic equations, with a linear diffusion term added to the momentum equation to account for heat conduction and viscosity, may be written as

$$\left(\tfrac{1}{2}u^{*2}+\frac{1}{\gamma-1}a^{*2}\right)_{x^*}+u^*_{t^*} = (\tfrac{4}{3}+\delta)\frac{\beta\mu_0^*}{\rho_0^*}u^*_{x^*x^*} \tag{3.1a}$$

and

$$\left(\tfrac{1}{2}u^{*2}+\frac{1}{\gamma-1}a^{*2}\right)_{t^*}+u^*\left(\tfrac{1}{2}u^{*2}+\frac{1}{\gamma-1}a^{*2}\right)_{x^*}-u^*u^*_{t^*}+(a^{*2}-u^{*2})u^*_{x^*} = 0. \tag{3.1b}$$

A function ϕ^* is now defined so that

$$\phi^*_{x^*} = u^*,$$
$$\phi^*_{t^*} = -\tfrac{1}{2}u^{*2}+(\tfrac{4}{3}+\delta)\frac{\mu_0^*\beta}{\rho_0^*}u^*_{x^*}+(a_0^{*2}-a^{*2})/(\gamma-1).$$

This function, which automatically satisfies the momentum equation (3.1a), is substituted into (3.1b) to yield

$$\phi^*_{t^*t^*}+2\phi^*_{x^*}\phi^*_{x^*t^*}-(a_0^{*2}-(\gamma-1)\phi^*_{t^*})\phi^*_{x^*x^*}-\frac{\beta\mu_0^*}{\rho_0^*}\phi^*_{x^*x^*t^*}$$
$$= \frac{\beta\mu_0^*}{\rho_0^*}(\phi^*_{x^*}\phi^*_{x^*x^*x^*}+(\gamma-1)\phi^{*2}_{x^*x^*}-\Gamma\phi^*_{x^*x^*}\phi^{*2}_{x^*}). \tag{3.2}$$

If higher-order non-linear diffusion terms on the right-hand side of (3.2) are dropped, and the resulting equation is expressed in non-dimensional acoustic co-ordinates x, t, we find

$$\phi_{tt}-\phi_{xx} = \epsilon(\beta\phi_{xxt}-2\phi_x\phi_{xt}-(\gamma-1)\phi_t\phi_{xx}), \tag{3.3}$$

where $\phi = \rho_0^*\phi^*/\epsilon\mu_0^*$. We shall refer to this equation as the *generalized Burgers equation.* It is convenient to introduce the quantities Q and u defined by

$$\epsilon Q = -\phi_{t^*}^*/a_0^{*2}$$

and

$$\epsilon u = \phi_{x^*}^*/a_0^*.$$

In terms of Q and u the generalized Burgers equation may be written as the system

$$u_t+Q_x = 0 \tag{3.3a}$$

and

$$Q_t+u_x = \epsilon(\beta Q_{xx}-2uQ_x-(\gamma-1)Qu_x). \tag{3.3b}$$

Using the fact that $a^{*2}/a_0^{*2} = T^*/T_0^* = 1+\epsilon T$ and the definition of ϕ, we see that Q is like a temperature in the acoustic region:

$$Q = \frac{1}{\gamma-1}T+\epsilon(\tfrac{1}{2}u^2-\beta u_x). \tag{3.4}$$

4. Shock reflexion from an adiabatic wall

We now apply the machinery of the theory of matched asymptotic expansions to construct an expansion in ϵ for the structure of a reflecting shock wave. Terms correct to $O(\epsilon)$ are obtained from the solution to the generalized Burgers equation by correcting the temperature with a formula derived from the Navier–Stokes equations.

The Burgers variables suitable for the study of right-going waves, ξ and $\bar{t}$, are chosen as the appropriate co-ordinates in which to view the problem when the incident shock is still many shock lengths from the wall. In terms of these co-ordinates the location of the wall is given by $\xi_w = -t/\epsilon$. For reasons which will become apparent shortly, these right-going Burgers co-ordinates are used with the restriction that $\bar{t} < 0$; after reflexion the appropriate co-ordinates are $\eta = x+t$ and $\bar{t}$. The wall location and shock path are sketched in figure 2 as they would appear in a $(x \pm t, \bar{t})$-plane.

We now focus our attention on the incident shock and seek an asymptotic expansion of the solution of the generalized Burgers equation written in right-going Burgers co-ordinates. In terms of these co-ordinates, equations (3.3) take the form

$$\bar{u}_\xi-\bar{Q}_\xi = \epsilon(\beta\bar{Q}_{\xi\xi}-\bar{Q}_{\bar{t}}-2\bar{u}\bar{Q}_\xi-(\gamma-1)\bar{Q}u_\xi) \tag{4.1}$$

and

$$\bar{u}_\xi-\bar{Q}_\xi = \epsilon\bar{u}_{\bar{t}}. \tag{4.2}$$

The first-order terms in the expansion are defined by

$$\bar{Q}^{(1)}(\xi,\bar{t}) = \overline{\lim}\,\bar{Q}(\xi,\bar{t};\epsilon)$$

and

$$\bar{u}^{(1)}(\xi,\bar{t}) = \overline{\lim}\,\bar{u}(\xi,\bar{t};\epsilon),$$

where the barred limit indicates that the right-going Burgers variables are to be fixed as $\epsilon\rightarrow 0$. Some care must be exercised in the application of the limiting process to equations (4.1) and (4.2) because the same equation results from both (4.1) and (4.2); i.e. $\overline{\lim}$ applied to either equation leads to the single equation

$$\bar{u}^{(1)}_{\xi} - \bar{Q}^{(1)}_{\xi} = 0. \tag{4.3}$$

To overcome this difficulty we simply set the right-side of (4.1) equal to the right side of (4.2) before taking $\overline{\lim}$. Thus we obtain the set of equations (4.2) and

$$\bar{u}_{\bar{t}} = \beta\bar{Q}_{\xi\xi} - \bar{Q}_{\bar{t}} - 2\bar{u}\bar{Q}_{\xi} - (\gamma-1)\bar{Q}\bar{u}_{\xi}. \tag{4.4}$$

The difficulty is now resolved because, when we apply $\overline{\lim}$ to equations (4.2) and (4.4), we obtain two independent equations for the two unknowns $\bar{Q}^{(1)}$ and $\bar{u}^{(1)}$. The same problem arises in deriving (2.3) from the Navier–Stokes equations, i.e. in taking the limit $\epsilon \to 0$ in equations (2.1) with ξ and $\bar{t}$ fixed.

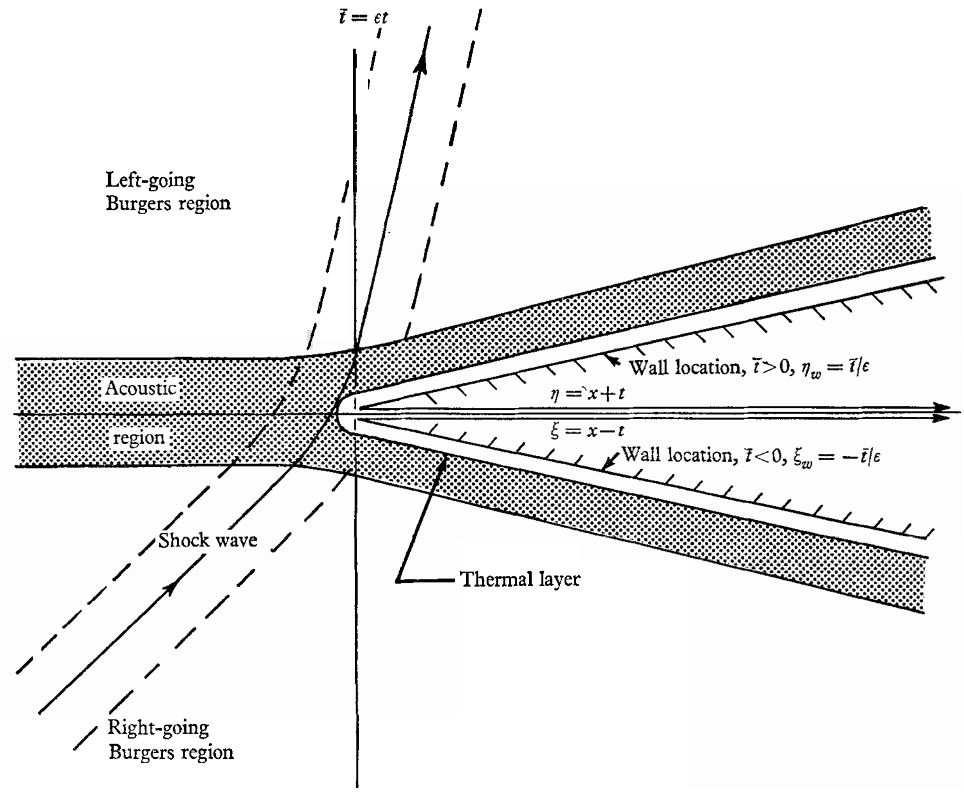

FIGURE 2. Wall location, shock-wave path and regions with different expansions in the $(x \pm t, \bar{t})$-plane.

The first-order terms in the right-going Burgers expansion satisfy the equations (4.3) and (4.4). The wall boundary condition is $\bar{u}(\xi_w, \bar{t}) = 0$, where $\xi_w \to \infty$ as $\epsilon \to 0$. This implies that $\bar{u}^{(1)}(\xi, \bar{t})$ must satisfy the condition $\bar{u}^{(1)}(\infty, \bar{t}) = 0$. The first-order form of the isentropic gas law is $\bar{T}^{(1)} = (\gamma-1)\bar{\rho}^{(1)}$, and equation (3.4) takes the first-order form $\bar{T}^{(1)} = (\gamma-1)\bar{Q}^{(1)}$. Thus we see that

$$\bar{Q}^{(1)} = \bar{\rho}^{(1)} = \bar{T}^{(1)}/(\gamma-1).$$

The above relations and the weak-shock limit of the conservation laws imply that $\bar{u}^{(1)}(-\infty,\bar{t}) = \bar{Q}^{(1)}(-\infty,\bar{t})$. Equation (4.3) may now be integrated; the last relation is used to evaluate the arbitrary function of $\bar{t}$ that arises in the integration. Thus we find that $\bar{u}^{(1)} = \bar{Q}^{(1)}$. Using this last result to eliminate $\bar{Q}^{(1)}$ from (4.4), we arrive at Burgers equation for right-going waves:

$$\bar{u}^{(1)}_{\bar{t}} + \Gamma\bar{u}^{(1)}\bar{u}^{(1)}_{\xi} = (\tfrac{1}{2}\beta)\bar{u}^{(1)}_{\xi\xi}. \tag{4.5}$$

In addition to the above equation and boundary condition, $u^{(1)}(\xi,\bar{t})$ must satisfy the initial condition that as t goes to minus infinity $\bar{u}^{(1)}$ should approach the Taylor weak-shock structure $f(\xi-\Gamma\bar{t}/2)$. The desired solution for $\bar{u}^{(1)}(\xi,\bar{t})$ which fills all the above requirements is $f(\xi-\Gamma\bar{t}/2)$ itself.

As the shock nears the wall, or equivalently as t goes to zero, the right-going Burgers expansion becomes invalid because it cannot take into account any disturbance that travels with a speed that is not nearly $u+a$: the wall has effectively zero speed and the reflected wave speed $-a$. If we take the acoustic limit of the generalized Burgers equation with definitions of the form

$$u^{(1)}(x,t) = \lim u(x,t;\epsilon)$$

we find that

$$\left.\begin{aligned} Q^{(1)}_t + u^{(1)}_x &= 0, \\ Q^{(1)}_x + u^{(1)}_t &= 0. \end{aligned}\right\} \tag{4.6}$$

and

Clearly $Q^{(1)}$ and $u^{(1)}$ satisfy the ordinary linear homogeneous wave equation. The acoustic limit of the wall boundary condition shows that $u^{(1)}(0,t) = 0$. The unique solution of the linear wave equation requires that we specify an initial condition. This condition is found by requiring that our acoustic expansion match the right-going Burgers expansion. In terms of the partial expansion symbols defined in §2, this matching condition takes the form

$$E_1\bar{u}^{(1)}(\xi,\bar{t}) = \bar{E}_1 u^{(1)}(x,t). \tag{4.7}$$

Consider the left side of this expression, which is a symbolic representation of a one-term acoustic expansion of the first term in the right-going Burgers expansion of the solution. To evaluate it, we first express the Burgers term in acoustic variables, and then find the first term in an expansion in ϵ:

$$E_1\bar{u}^{(1)}(\xi,\bar{t}) = E_1 f(\xi-\Gamma\bar{t}/2) = E_1 f(x-t-\epsilon\Gamma t/2) = f(x-t).$$

The right side of (4.7) is the limit as $\epsilon\to 0$ of the first term in the acoustic expansion for u, expressed in right-going Burgers co-ordinates. This can be interpreted as a condition on $u^{(1)}$ as follows:

$$\bar{E}_1 u^{(1)}(x,t) = \overline{\lim}\, u^{(1)}(\xi+\bar{t}/\epsilon, \bar{t}/\epsilon).$$

Another way of expressing this last result is to note that letting $\epsilon\to 0$ is equivalent to letting t go to minus infinity with $\xi = x-t$ kept constant as required by $\overline{\lim}$. We can now write the initial condition to be imposed on $u^{(1)}$ as the requirement that as $t\to -\infty$ with $x-t$ fixed, $u^{(1)}(x,t)\to f(x-t)$.

As the solution of equations governing higher-order terms in the acoustic expansion requires the ability to solve inhomogeneous wave equations, we will need the solution of the problem:

$$u_{tt} - u_{xx} = H(x-t, x+t) \tag{4.8}$$

with

$$u(0,t) = 0,$$

and as $t \to -\infty$ with $x-t$ fixed, $u(x,t) \to u_1(x-t)t^{k-1}$. The solution to (4.8) is most easily expressed in the characteristic co-ordinates† $\xi = x-t$ and $\eta = x+t$:

$$u(\xi,\eta) = u(\xi,\eta_1) - u(-\eta,\eta_1) - \int_{\eta_1}^{\eta}\int_{\xi}^{-\eta} H(\xi,\eta)\,d\xi\,d\eta. \tag{4.9}$$

The initial conditions are applied on the characteristic line $\eta = \eta_1$. For our problem the initial conditions are actually limiting conditions as t goes to minus infinity. Because the co-ordinate ξ is fixed in this limiting process, $t \to -\infty$ implies that $\eta \to -\infty$. Thus to use formula (4.9) for our problem, we apply the condition at a fixed η which we call η_1, and then let η_1 go to $-\infty$ in the resulting expression. Because of the inhomogeneous term, there is a term in the integral part of (4.9) which cancels the growth terms occurring in this limiting operation. Hence we see that it is the growth terms that are responsible for the non-uniformity of the acoustic expansion and make it possible to match the Burgers expansion to the acoustic expansion.

In the case of the first-order acoustic problem $k = 1$, $u_1 = f(\xi)$, $u = u^{(1)}$ and $H \equiv 0$, so that the above formula gives us the result

$$u^{(1)} = f(\xi) - f(-\eta). \tag{4.10}$$

The remaining first-order acoustic variables $\rho^{(1)}$, $T^{(1)}$ are found by using the isentropic gas law for vanishing ϵ, and equation (4.6) for $Q^{(1)}$. The results are:

$$\rho^{(1)} = f(\xi) + f(-\eta),$$

and

$$T^{(1)} = (\gamma-1)[f(\xi) + f(-\eta)].$$

This completes the first-order acoustic solution. As we have already noted in §2, the acoustic expansion is not uniformly valid for large time. To obtain a valid expansion for large positive time we must use a left-going Burgers expansion, i.e. an asymptotic expansion of the solution written in the co-ordinates $\eta, \bar{t}$. Thus we express the generalized Burgers equation in these co-ordinates and let $\epsilon \to 0$ with η and $\bar{t}$ fixed. With definitions of the type

$$\bar{u}^{(1)}(\eta,\bar{t}) = \overline{\lim}\, \bar{u}(\eta,\bar{t};\epsilon),$$

and the conditions that $\bar{u}^{(1)} = \bar{Q}^{(1)} = 1$ at $\eta = -\infty$, the process $\overline{\lim}$ results in Burgers equation for left-going waves,

$$\bar{u}^{(1)}_{\bar{t}} + (\Gamma\bar{u}^{(1)} - \gamma + 1)\bar{u}^{(1)}_{\eta} = (\tfrac{1}{2}\beta)\bar{u}^{(1)}_{\eta\eta}.$$

The variables $\bar{\rho}^{(1)}(\eta,\bar{t})$ and $\bar{T}^{(1)}(\eta,\bar{t})$ are found in a straightforward manner to be

$$\bar{\rho}^{(1)}(\eta,\bar{t}) = 2 - \bar{u}^{(1)}(\eta,\bar{t})$$

and

$$\bar{T}^{(1)}(\eta,\bar{t}) = (\gamma-1)\bar{\rho}^{(1)}(\eta,\bar{t}).$$

The requirement that the left-going solution match the acoustic solution determines the initial condition that $\bar{u}(\eta,\bar{t})$ must satisfy. By analogy with our matching of the right-going Burgers expansion with the acoustic expansion, one

† These co-ordinates will be used interchangeably with (x, t) for the acoustic co-ordinates and should not be confused with the $(\xi, \bar{t})$ or $(\eta, \bar{t})$ co-ordinate systems.

can see that the matching condition is equivalent to the requirement that as $\bar{t}\to 0$, $\bar{u}^{(1)}(\eta,\bar{t})\to f(\eta)$. Thus we find that the appropriate solution to (4.8) is

$$\bar{u}^{(1)}(\eta,\bar{t}) = f\left(\eta - \frac{5-3\gamma}{4}\bar{t}\right). \tag{4.11}$$

We now return to the acoustic region to find the second term in the acoustic expansion of the solution to the generalized Burgers equation. By using the correction formula derived in the appendix for the second-order temperature, we may use the generalized Burgers equation to derive the second-order acoustic expansion that would be obtained from the full Navier–Stokes equations. G.I. Taylor observed that the structure of a shock wave is maintained by a balance between convective and diffusive effects. This balance cannot be maintained near the wall because the convective term vanishes at the wall. However, the acoustic solution shows that these interesting effects are second-order in ϵ. To assess their influence, we must determine the second-order term in the acoustic expansion. The equations governing the second-order acoustic terms are found by substituting the series

$$Q(x,t;\epsilon) = Q^{(1)}(x,t) + \epsilon Q^{(2)}(x,t) + O(\epsilon^2)$$

and

$$u(x,t;\epsilon) = u^{(1)}(x,t) + \epsilon u^{(2)}(x,t) + O(\epsilon^2)$$

into the generalized Burgers equation and using the results (4.6). Thus we find the equations

$$\left.\begin{aligned} u_t^{(2)} + Q_x^{(2)} &= 0, \\ u_t^{(2)} + Q_x^{(2)} &= \beta Q_{xx}^{(1)} - 2u^{(1)}Q_x^{(1)} - (\gamma-1)Qu_x, \end{aligned}\right\} \tag{4.12}$$

and the auxiliary relations

$$\left.\begin{aligned} Q^{(2)} &= \frac{1}{\gamma-1}T^{(2)} + \tfrac{1}{2}u^{(1)2} - \beta u_x^{(1)} \\ \rho^{(2)} &= \frac{1}{\gamma-1}T^{(2)} + \frac{2-\gamma}{2(\gamma-1)^2}T^{(1)2}. \end{aligned}\right\} \tag{4.13}$$

The last of the equations (4.13) is obtained from the isentropic gas law.

The second-order acoustic expansion of the generalized Burgers equation is governed by the system (4.12). With $u^{(2)}(x,t)$ and $Q^{(2)}(x,t)$ determined, the relations (4.13) prescribe the second-order density and temperature. In the appendix we compare the first- and second-order acoustic expansion of the Navier–Stokes equations with (4.6), (3.4), (4.12) and (4.13). This comparison shows that the acoustic expansion of the generalized Burgers equation gives the velocity and density to second order, but that the second-order temperature, $T_{NS}^{(2)}$, satisfies

$$(T_{NS}^{(2)})_t = T_t^{(2)} + \frac{\gamma}{Pr}T_{xx}^{(1)} \tag{4.14}$$

with $T^{(2)}(x,t)$ determined from (4.12) and (4.13). Equation (4.14) then gives, after a quadrature, the second-order temperature $T_{NS}^{(2)}$. Thus (4.12) and (4.13), coupled with the correction formula (4.14), determine the second-order acoustic expansion of the Navier–Stokes equations.

The boundary condition that must be satisfied by $u^{(2)}(0,t)$, $x = 0$ is clearly $u^{(2)}(0,t) = 0$. Initial conditions are supplied by the matching principle; that is, we require $E_2\bar{E}_1\bar{u} = \bar{E}_1 E_2 u$. This is equivalent to requiring that as $t \to -\infty$ with $x-t$ fixed, $u^{(2)}(x,t) \to \Gamma f'(x-t)t/2$. Transforming the inhomogeneous wave equation satisfied by $u^{(2)}$ into acoustic characteristic co-ordinates, and substituting (4.10) for $u^{(1)}$, we find

$$u^{(2)}_{\xi\eta} = \left(-\frac{\beta}{4}f''(\eta) - \frac{\Gamma}{2}f(-\eta)f'(\eta) + \frac{3-\gamma}{4}f(\xi)f'(\eta)\right)_\eta + \left(\frac{\beta}{4}f''(\xi) - \frac{\Gamma}{2}f(\xi)f'(\xi) + \frac{3-\gamma}{4}f(-\eta)f'(\xi)\right)_\xi.$$

Thus the second-order acoustic velocity is determined by a Goursat problem of the type discussed earlier. Using formula (4.9), we find that

$$u^{(2)} = \frac{\Gamma t}{2}(f'(\eta) - f'(\xi)) + \frac{3-\gamma}{4}\frac{\beta}{\Gamma}(f'(\eta)\log f(-\xi) - f'(\xi)\log f(\eta)).$$

To find the other second-order variables we need only substitute the above solution for $u^{(2)}$ into equations (4.12) and (4.13). The arbitrary function of x introduced by the integration is evaluated by imposing the condition that the second-order solution vanish as t goes to minus infinity with x fixed. The second-order terms in the acoustic expansion for ρ, T and p are found to be

$$\rho^{(2)} = f^2(-\eta) + f^2(\xi) + \tfrac{1}{2}\Gamma(f(\eta) - f(\xi) - 1) - \tfrac{1}{2}\Gamma t(f'(\xi) + f'(\eta)) - \frac{3-\gamma}{4}\frac{\beta}{\Gamma}(f'(\xi)\log f(\eta) + f'(\eta)\log f(-\xi)) + \frac{3-\gamma}{2}f(-\eta)f(\xi),$$

$$T^{(2)}_{NS}/(\gamma-1) = \rho^{(2)} + \frac{\gamma}{Pr}(f'(\xi) + f'(\eta)) - \frac{2-\gamma}{2}(f(\xi) + f(-\eta))^2$$

and

$$p^{(2)} = \rho^{(2)} + T^{(2)} + (\gamma - 1)(f(\xi) + f(-\eta))^2.$$

We can verify that the second-order acoustic solution matches the left-going Burgers expansion. Thus one finds that

$$\bar{E}_1 E_2 u = E_2 \bar{E}_1 u = f(\eta) - \epsilon\frac{5-3\gamma}{4}tf'(\eta)$$

and

$$\bar{E}_1 E_2 \rho = E_2 \bar{E}_1 \rho = 2 - f(\eta) + \epsilon\frac{5-3\gamma}{4}tf'(\eta).$$

It is possible to construct composite expansions by several different procedures. We employ the additive composition principle of Van Dyke (1964). These expansions are constructed so that they reduce to the Burgers or the acoustic expansion in the appropriate limit. The procedure is this: consider a function $W(x,t)$ which may represent any of the variables of interest, and form the composite expansion

$$W_{\text{comp}} = \bar{E}_1 W + E_2 W - \bar{E}_1 E_2 W.$$

The composite expansion for u is given by

$$u_{\text{comp}} = f(\xi - \epsilon\Gamma t/2) - f(-\eta) + \epsilon\left[\Gamma t f'(\eta)/2 + \frac{3-\gamma}{4}\frac{\beta}{\Gamma}(f'(\eta)\log f(-\xi) - f'(\xi)\log f(\eta))\right] \tag{4.14a}$$

for $t \leqslant 0$, and

$$u_{\text{comp}} = f\left(\eta - \frac{5-3\gamma}{4}\epsilon t\right) + f(\xi) - 1 + \epsilon\left[\frac{3-\gamma}{2} t f'(\eta) - \frac{\Gamma}{2} t f'(\xi) + \frac{3-\gamma}{4}\frac{\beta}{\Gamma}(f'(\eta)\log f(-\xi) - f'(\xi)\log f(\eta))\right] \tag{4.14b}$$

for $t \geqslant 0$.

Composite representations of the other dependent variables may be found in the same way.

Before discussing some of the implications of the results, we first consider the effect of an isothermal wall.

5. The isothermal wall

In the last section we determined the structure of a shock wave undergoing reflexion from an adiabatic wall. The boundary condition on temperature was that its gradient $T_x(0,t)$ vanish at the wall. This condition is automatically satisfied for the acoustic equations if the gas velocity vanishes there. Consequently, the first-order acoustic equations are degenerate at the wall because they are unable to satisfy an arbitrary temperature boundary condition. This state of affairs again signals for the method of matched asymptotic expansions; hence we seek a co-ordinate stretching that leads to a valid expansion in the wall region. The expansion of the solution near the wall, i.e. in the thermal layer, must then be matched with the acoustic expansion obtained previously. The generalized Burgers equation used in the above analysis is unable to cope with an arbitrary temperature boundary condition. For this reason we must return to the Navier–Stokes equations (2.1) for our analysis. As pointed out by Goldsworthy (1959), the pressure gradient in a thermal layer of the type we are considering should nearly vanish. This provides a check on our co-ordinate stretching, i.e. we expect the equations governing the first-order terms in a thermal layer expansion to indicate the absence of a pressure gradient.

The thermal layer co-ordinates are defined by the stretching $\hat{x} = \epsilon^{-\frac{1}{2}}x$, $\hat{t} = t$. To find the form of the thermal layer expansion, we use the matching principle $\hat{E}_1 E_1 u = E_1 \hat{E}_1 u$, where $\hat{E}_n$ indicates the n term thermal layer expansion. Consider the thermal layer expansion of the first-order acoustic velocity:

$$\begin{aligned}\hat{E}_1 E_1 u &= \hat{E}_1 u^{(1)}(x,t) = \hat{E}_1[f(x-t) - f(-x-t)] \\ &= \hat{E}_1[f(\epsilon^{\frac{1}{2}}\hat{x} - \hat{t}) - f(-\epsilon^{\frac{1}{2}}\hat{x} - \hat{t})] \\ &= 2\hat{x}f'(-\hat{t})\epsilon^{\frac{1}{2}}.\end{aligned}$$

This shows that the thermal layer velocity expansion must have the form

$$\hat{u}(\hat{x},\hat{t};\epsilon) = \epsilon^{\frac{1}{2}}\hat{u}^{(1)}(\hat{x},\hat{t}) + O(\epsilon)$$

in order to match with the acoustic expansion. Because $x \to 0$ when $\epsilon \to 0$ with $\hat{x}$ fixed, and $\hat{x} \to -\infty$ when $\epsilon \to 0$ with x fixed, the condition imposed on the first-order thermal layer solution by the matching condition is equivalent to

$$\lim_{\hat{x} \to -\infty} \hat{u}^{(1)}(\hat{x}, \hat{t}) = 2\hat{x}f'(-\hat{t}).$$

Similar considerations show that the thermal layer expansion of $\hat{\rho}$, and hence $\hat{p}$ and $\hat{T}$, must be of the form $\rho = \hat{\rho}^{(1)} + o(\hat{\rho}^{(1)})$. Substitution of such expansions into the Navier–Stokes equations expressed in thermal co-ordinates leads to the following set of equations for the first-order thermal layer variables:

$$\left.\begin{aligned} &\hat{u}^{(1)}_{\hat{x}} + \hat{\rho}^{(1)}_{\hat{t}} = 0, \\ &\hat{p}^{(1)}_{\hat{x}} = 0, \\ &\hat{T}^{(1)}_{\hat{t}} + (\gamma - 1)\hat{u}^{(1)}_{\hat{x}} - \frac{\gamma}{Pr}\hat{T}^{(1)}_{\hat{x}\hat{x}} = 0 \\ \text{and} \qquad &\hat{\rho}^{(1)} + \hat{T}^{(1)} = \hat{p}^{(1)}. \end{aligned}\right\} \tag{5.1}$$

The wall boundary conditions are

$$\hat{T}^{(1)}(0, \hat{t}) = \hat{u}^{(1)}(0, \hat{t}) = 0,$$

and the matching principle leads to the following boundary conditions at $x = -\infty$:

$$\left.\begin{aligned} &\hat{u}^{(1)} = 2f'(-\hat{t})\hat{x}, \\ &\hat{\rho}^{(1)} = 2f(-\hat{t}) \\ \text{and} \qquad &\hat{T}^{(1)} = 2(\gamma - 1)f(-\hat{t}). \end{aligned}\right\} \tag{5.2}$$

Using the second and fourth of equations (5.1) and conditions at $x = -\infty$, we find that

$$\hat{p}^{(1)} = 2\gamma f(-\hat{t}).$$

A simple manipulation of the thermal equations shows that the first-order thermal layer temperature satisfies the inhomogeneous heat equation

$$\hat{T}^{(1)}_{\hat{t}} - \frac{1}{Pr}\hat{T}^{(1)}_{\hat{x}\hat{x}} = -2(\gamma - 1)f'(\hat{t}). \tag{5.3}$$

If we define

$$\hat{\theta} = \hat{T}^{(1)} - 2(\gamma - 1)f(-\hat{t}),$$

then equation (5.3) becomes

$$\hat{\theta}_{\hat{t}} - \frac{1}{Pr}\hat{\theta}_{\hat{x}\hat{x}} = 0,$$

with

$$\hat{\theta}(0, \hat{t}) = -2(\gamma - 1)f(-\hat{t}).$$

The solution of this last problem is straightforward (see, for example, Tychonov & Samarski 1964). We find

$$\hat{\theta}(\hat{x}, \hat{t}) = -\frac{2(\gamma - 1)}{\pi^{\frac{1}{2}}}\int_0^\infty \frac{e^{-z}}{z^{\frac{1}{2}}} f\left(\frac{\hat{x}^2 Pr}{4z} - \hat{t}\right) dz.$$

In terms of this result the first-order temperature, density and velocity in the thermal layer are

$$\hat{T}^{(1)} = \hat{\theta} + 2(\gamma - 1)f(-\hat{t}),$$

$$\hat{\rho}^{(1)} = \hat{p}^{(1)} - \hat{T}^{(1)} = 2f(-\hat{t}) - \hat{\theta}$$

and

$$\hat{u}^{(1)} = -\int_0^{\hat{x}} \hat{\rho}^{(1)}_{\hat{t}}(\hat{x}', \hat{t})\, d\hat{x}'.$$

The influence of the isothermal wall on the solution in the acoustic region can be seen by expanding the above solution, written in acoustic co-ordinates, in ϵ. Thus we find, e.g. that

$$E_2\hat{T}^{(1)} = 2(\gamma-1)f(-t)+O(e^{x/\epsilon^{\frac{1}{2}}}),$$

but that

$$E_2\epsilon^{\frac{1}{2}}\hat{u}^{(1)}(\hat{x},\hat{t}) = 2f'(-t)x+\epsilon^{\frac{1}{2}}\frac{2(\gamma-1)}{\pi^{\frac{1}{2}}}\int_0^\infty\int_0^\infty \frac{e^{-z}}{z^{\frac{1}{2}}}f'\left(\frac{\alpha^2 Pr}{4z}-t\right)dz\,d\alpha+O(\epsilon^{\frac{1}{2}}e^{x/\epsilon^{\frac{1}{2}}}). \tag{5.4}$$

Because the second-order terms in the acoustic expansions of $\hat{\rho}^{(1)}$ and $\hat{T}^{(1)}$ are exponentially small, these terms can have no effect on the boundary conditions for the second-order acoustic expansion. However, the second-order term in the acoustic expansion of $\hat{u}^{(1)}$ is $O(\epsilon^{\frac{1}{2}})$. This implies that the acoustic and thermal layer expansions can only be matched if the acoustic expansion takes the form

$$u(x,t) = u^{(1)}+\epsilon^{\frac{1}{2}}u^{(2)}+O(\epsilon).$$

Substitution of this and of similar expansions for the thermodynamic variables into the acoustic form of the Navier–Stokes equations shows that $u^{(2)}$ satisfies the homogeneous wave equation. It is easy to show that the new wall boundary conditions implied by (5.4) do not affect the right-going Burgers expansion. Using the initial condition provided by the acoustic expansion of the right-going Burgers expansion, we see that as $t\to-\infty$ with $x-t$ fixed, $u^{(2)}\to 0$. We then find that the solution which satisfies this initial condition and the time-dependent wall boundary condition obtained from (5.4) is

$$u^{(2)}(x,t) = \frac{2(\gamma-1)}{\pi^{\frac{1}{2}}}\int_0^\infty\int_0^\infty \frac{e^{-z}}{z^{\frac{1}{2}}}f'\left(\frac{\alpha^2 Pr}{4z}-x-t\right)dz\,d\alpha. \tag{5.5}$$

The matching condition $E_2\bar{E}_1u = \bar{E}_1E_2u$ shows that the thermal layer has an exponentially small effect on the right-going Burgers expansion, but requires a term of $O(\epsilon^{\frac{1}{2}})$ in the left-going Burgers expansion:

$$\bar{u}(\eta,\bar{t}) = \bar{u}^{(1)}(\eta,\bar{t})+\epsilon^{\frac{1}{2}}\bar{u}^{(2)}(\eta,\bar{t}).$$

Here the matching condition supplies the initial condition

$$\bar{u}(\eta,0) = f(\eta)+\epsilon^{\frac{1}{2}}U(\eta), \tag{5.6}$$

where $\epsilon^{\frac{1}{2}}U(\eta)$, an 'effective wall velocity' that accounts for the displacement effect due to the presence of the thermal layer, is given by (5.5):

$$U(\eta) = \frac{\gamma-1}{2}\left(\frac{2\Gamma}{\pi\beta Pr}\right)^{\frac{1}{2}}\int_0^\infty \alpha^{-\frac{1}{2}}\operatorname{sech}^2(\alpha-\Gamma\eta/2\beta)\,d\alpha. \tag{5.7}$$

It is also convenient to introduce here the 'effective wall displacement', $\epsilon^{\frac{1}{2}}X(\eta)$, where

$$X(\eta) = 2(\gamma-1)\left(\frac{2\beta}{\pi\Gamma Pr}\right)^{\frac{1}{2}}\int_0^\infty \alpha^{\frac{1}{2}}\operatorname{sech}^2(\alpha-\Gamma\eta/2\beta)\,d\alpha.$$

Thus the structure of our weak shock wave after it reflects from an isothermal wall is the same as the structure of a weak shock wave after it reflects from an adiabatic wall whose location is prescribed by $\epsilon^{\frac{1}{2}}X(\bar{t}/\epsilon)$.

Goldsworthy (1959) computed the effect of a heat conducting wall on the trajectory of the reflected shock wave by computing the displacement effect of the thermal boundary layer near the wall. His result for the effective wall velocity becomes

$$\epsilon^{\frac{1}{2}}U(t) = \epsilon^{\frac{1}{2}}2(\gamma-1)/(\pi Prt)^{\frac{1}{2}}$$

when the weak shock approximations are made. This is simply the asymptotic behaviour of our expression (5.7). Goldsworthy's thermal boundary-layer analysis correctly predicts the asymptotic behaviour of the reflected shock trajectory, and our analysis provides the structure of the reflecting shock wave as long as it is weak. Figure 3 displays the effective wall velocity and displacement as a function of time, as well as their asymptotic behaviour.

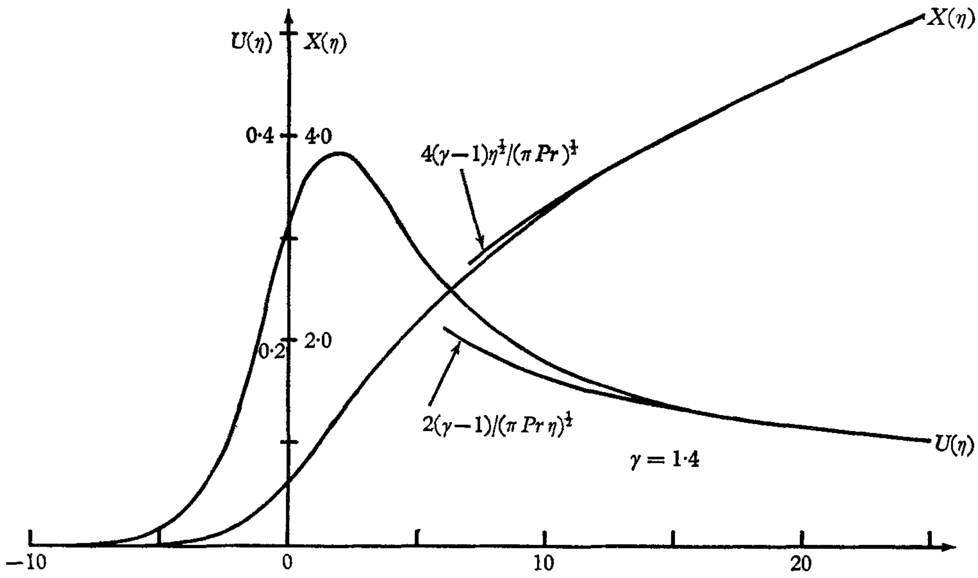

FIGURE 3. Effective wall velocity and displacement due to the thermal layer.

To compute the structure of the reflected wave we need to solve the system

$$\bar{u}_\eta + \bar{Q}_\eta + \epsilon \bar{u}_{\bar{t}} = 0,$$

$$\bar{Q}_{\bar{t}} - \bar{u}_{\bar{t}} = \beta \bar{Q}_{\eta\eta} - 2\bar{u}\bar{Q}_\eta - (\gamma-1)\bar{Q}\bar{u}_\eta,$$

subject to the initial condition (5.6) and the boundary condition that as $\eta \to -\infty$, $\bar{u}(\eta, \bar{t}) \to 1$. Because the initial conditions, and consequently the next order solution we are seeking, contain terms $O(1)$ and $O(\epsilon^{\frac{1}{2}})$, while the equations contain terms $O(1)$ and $O(\epsilon)$, it is clear that we may obtain simultaneously the first two terms,

$$u^{(1)} + \epsilon^{\frac{1}{2}}\bar{u}^{(2)},$$

by solving the equation

$$\bar{u}_{\bar{t}} + (\Gamma u - \gamma + 1)\bar{u}_\eta = \tfrac{1}{2}\beta \bar{u}_{\eta\eta} \tag{5.8}$$

subject to the initial condition (5.6) and the boundary condition $\bar{u}(-\infty,\bar{t}) = 1$. This procedure appears to involve fewer analytical complications than that of solving the linear equation for $\bar{u}^{(2)}$ with $\bar{u}^{(1)}$ given by (4.11). A Hopf–Cole transformation of the form

$$\phi(\eta,\bar{t}) = \exp\left\{-\frac{(3-\gamma)^2}{8\beta}\bar{t}-\frac{3-\gamma}{2\beta}\eta-\frac{\Gamma}{\beta}\int_{-\infty}^{\eta}[\bar{u}(\alpha,\bar{t})-1]\,d\alpha\right\}$$

reduces (5.8) to the heat equation

$$\phi_{\bar{t}} = (\tfrac{1}{2}\beta)\phi_{\eta\eta},$$

subject to the initial condition

$$\phi(\eta,0) = \exp\left\{-\frac{3-\gamma}{2\beta}\eta-\frac{\Gamma}{\beta}\int_{-\infty}^{\eta}[f(\alpha)-1+\epsilon^{\frac{1}{2}}U(\alpha)]\,d\alpha\right\}.$$

It is then a simple matter to write a formal expression for $\phi(\eta,\bar{t})$. Once this is done, the velocity may be determined from the relation

$$\bar{u}(\eta,\bar{t}) = -\frac{\beta}{\Gamma}\frac{\phi_\eta}{\phi}+\frac{\gamma-1}{\Gamma}.$$

Here we have imposed the condition that $\bar{u}(-\infty,\bar{t}) = 1$.

The formal representation of $\bar{u}(\eta,\bar{t})$ may be written in the form

$$\bar{u}(V,\bar{t}) = \tfrac{1}{2}+\frac{V}{\Gamma}-\frac{\beta}{\Gamma\bar{t}}\frac{\partial}{\partial V}\left\{\log\int_{-\infty}^{\infty}\cosh\left(\frac{\Gamma z}{2\beta}\right)\exp\left[-\frac{z^2}{2\beta\bar{t}}-\frac{\epsilon^{\frac{1}{2}}\Gamma}{\beta}X(z)+\frac{Vz}{\beta}\right]dz\right\}, \tag{5.9}$$

where

$$V = \frac{\eta}{\bar{t}}-\frac{5-3\gamma}{4}.$$

When $\epsilon = 0$, this expression reduces to $\bar{u}(V,\bar{t}) = f(V\bar{t})$, and we note that the location of the midpoint of the velocity profile, $\bar{u} = \frac{1}{2}$, is given by $V = 0$.

To determine how the asymptotic structure of the shock wave has been affected by the isothermal wall, it is necessary to investigate the behaviour of (5.9) as $\bar{t}\to\infty$. Because we are interested in all values of V, the analytical details become cumbersome and we give only the result. We find

$$\bar{u}(\eta,\bar{t}) \sim 2(\gamma-1)\left(\frac{\epsilon}{\pi Pr}\right)^{\frac{1}{2}}[\eta+(\gamma-1)\bar{t}]^{-\frac{1}{2}}$$

for $\eta > \dfrac{3-\gamma}{2}\bar{t}$,

$$\bar{u}(\eta,\bar{t}) \sim f\left(\eta-\frac{5-3\gamma}{4}\bar{t}-4(\gamma-1)\left(\frac{\epsilon}{\pi Pr}\right)^{\frac{1}{2}}(\eta+(\gamma-1)\bar{t})^{\frac{1}{2}}\right)$$
$$\times\left\{1+2(\gamma-1)\left(\frac{\epsilon}{\pi Pr}\right)^{\frac{1}{2}}\frac{\exp\left[\eta-\frac{5-3\gamma}{4}\bar{t}-4(\gamma-1)\left(\frac{\epsilon}{\pi Pr}\right)^{\frac{1}{2}}(\eta+(\gamma-1)\bar{t})^{\frac{1}{2}}\right]}{[\eta+(\gamma-1)\bar{t}]^{\frac{1}{2}}}\right\}$$

for $\dfrac{3-\gamma}{2}\bar{t} > \eta > -(\gamma-1)\bar{t}$,

and
$$\bar{u}(\eta,\bar{t}) \sim 1$$

for $\eta < -(\gamma-1)\bar{t}$. We can use the second of these results to obtain the asymptotic shock location:

$$x_{\text{shock}} \sim -t + \frac{5-3\gamma}{4}\epsilon t + 2(\gamma-1)\epsilon\left(\frac{2\Gamma}{\pi Pr}\right)^{\frac{1}{2}} t^{\frac{1}{2}}.$$

The first two terms in the asymptotic expression represent the position of the shock wave were it reflected from an adiabatic wall. Consequently, a shock wave reflected from an isothermal wall travels at a velocity that is less than that for a shock wave reflected from an adiabatic wall; this decrement is asymptotic to $a_0^*(\gamma-1)(2\Gamma\epsilon/\pi t Pr)^{\frac{1}{2}}$, which is in agreement with the results of Goldsworthy (1959) and Sturtevant & Slachmuylders (1964).

6. Discussion and concluding remarks

We have employed the technique of matched asymptotic expansions to obtain an asymptotic representation of the structure of a weak shock wave undergoing reflexion from a wall. The results of our analyses are given in §§4 and 5. Here we will discuss some of the more interesting features of these results and some possible extensions of our procedures.

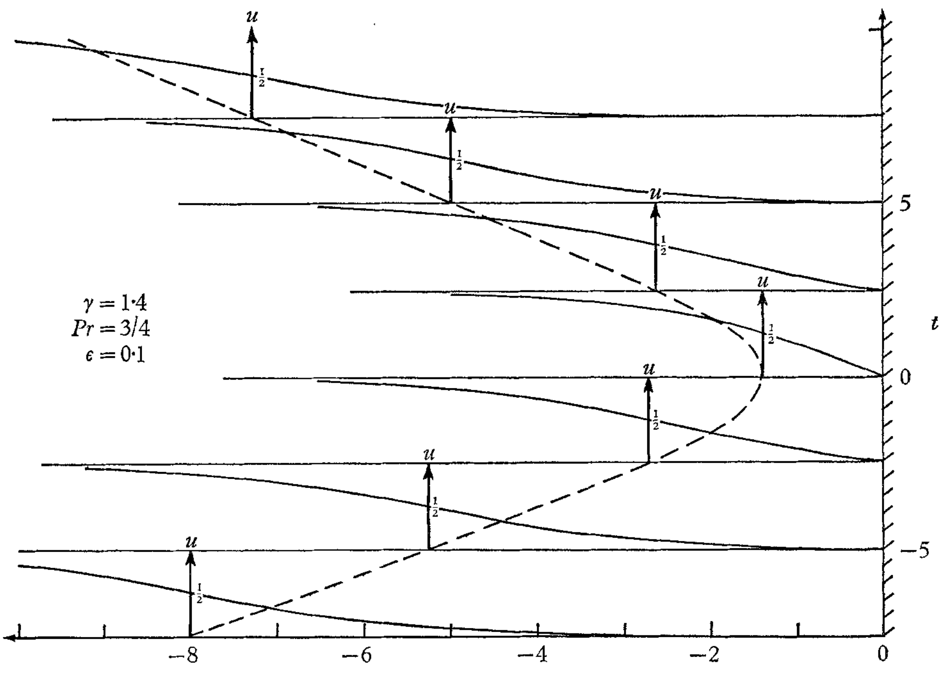

FIGURE 4. Velocity profiles as given by the composite expansion for an adiabatic wall. Dotted line is the locus of point in the (x, t)-plane where $u = \frac{1}{2}$.

Figure 4 shows several velocity profiles constructed from the composite expansion for the adiabatic wall problem. Near the wall the gas velocity must vanish and the balance between convection and diffusion that maintains the profile of a shock wave moving in a boundary free environment cannot continue to hold in the wall region. The first-order solution in this region satisfies the equations of linear acoustics. This solution is also the first-order solution for the interaction of two weak shock waves of equal strength, but moving in opposite directions. For the first-order solution the velocity reaches its maximum value at all x when $t = 0$. Thus an observer downstream of the shock could, if equipped with the appropriate instruments, detect the instant of shock reflexion by determining when the velocity at his station attained a maximum. The reason this behaviour prevails for the first-order solution is that solutions of the linear wave

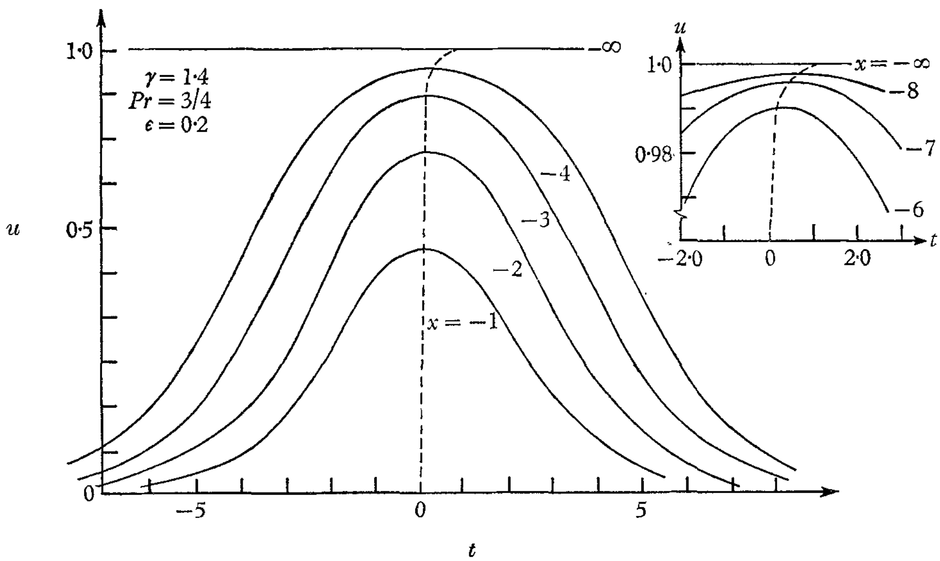

FIGURE 5. Velocity versus time at various x stations for an adiabatic wall. Dotted lines indicate the locus of points where u is a maximum for fixed x.

equation preserve their shape as they propagate. The linear solution to the wall problem consists of a positive velocity wave travelling to the right plus the reflexion of this wave travelling to the left, so that the maximum velocity occurs simultaneously at all x. Although steady state shock waves move at a speed in excess of the acoustic speed they also preserve their shape structure because the non-linear terms in the Navier–Stokes equations are balanced by the viscous terms. Near the wall this balance is offset to second order. This effect can be seen from the curves of velocity versus time at given x shown in figure 5. The dashed line is a locus of maximum velocity points; one can see that it takes a finite time for the velocity maximum to propagate upstream. It is well known that a shock wave becomes thinner as it becomes stronger. Figure 6, which compares velocity profiles obtained from the first- and second-order expansions, shows the increase in strength associated with the second-order terms. Figure 7 is a plot of temperature, density and pressure during reflexion, constructed from the composite solution for the adiabatic wall.

Our isothermal wall analysis shows that, in the cases of most practical interest, the structure and trajectory at the reflected shock are affected to $O(\epsilon^{\frac{1}{2}})$ by the wall temperature condition. For an isothermal wall these effects dominate the effects of viscous dissipation. This is to be expected because of the displacement effect at the thermal layer next to the wall boundary. Our results are in agreement with the results of Goldsworthy (1959) based on a boundary layer analysis of the

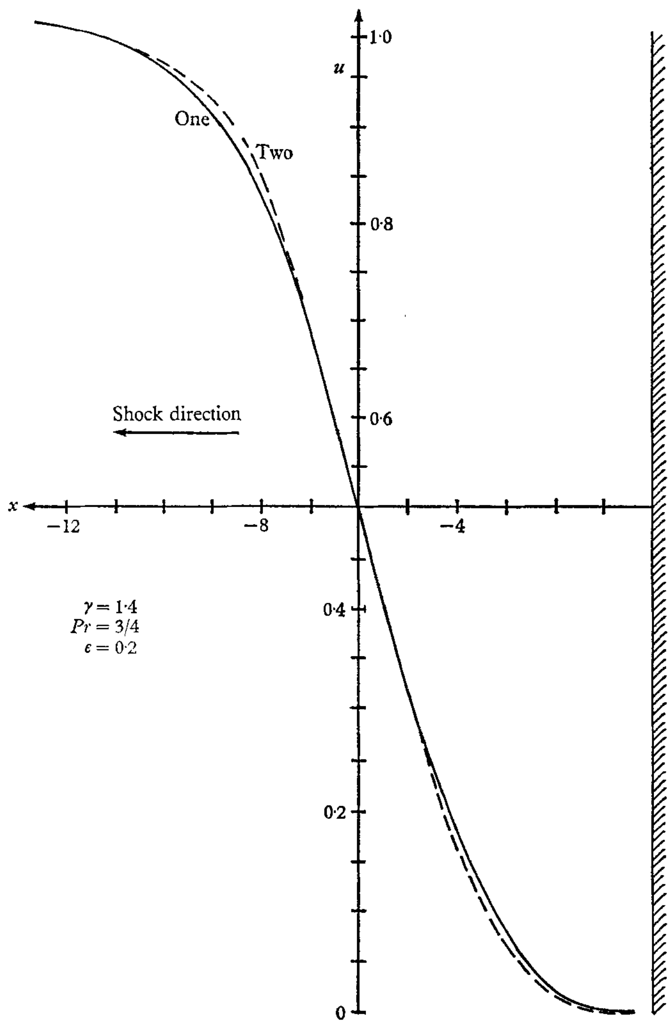

FIGURE 6. One- and two-term composite expansions for the velocity at $t = 6{\cdot}0$. (Adiabatic wall.)

thermal layer. In figure 8 we display the values of the thermodynamic variables at the wall as a function of time. For the isothermal wall

$$\hat{T}^{(1)}(0,\hat{t}) = 0 \quad \text{and} \quad \hat{p}^{(1)}(0,\hat{t}) = p^{(1)}(0,\hat{t}).$$

However, since $\hat{p}^{(1)} = \hat{\rho}^{(1)} + \hat{T}^{(1)}$ the density at the isothermal wall is the same as the pressure at the adiabatic wall. The variations of temperature and velocity through the thermal layer are shown in figure 9.

As noted earlier, we have integrated the generalized Burgers equation numerically. Our procedure employed an explicit finite difference technique similar to the one suggested by Richtmeyer (1957) for the coupled linear equations of sound and heat flow. The numerical solution for u was compared with our asymptotic expansion for $\epsilon = 0{\cdot}025$, $0{\cdot}05$ and $0{\cdot}10$. It was found that the theoretical and numerical solutions agreed to the order expected from the theoretical results.†

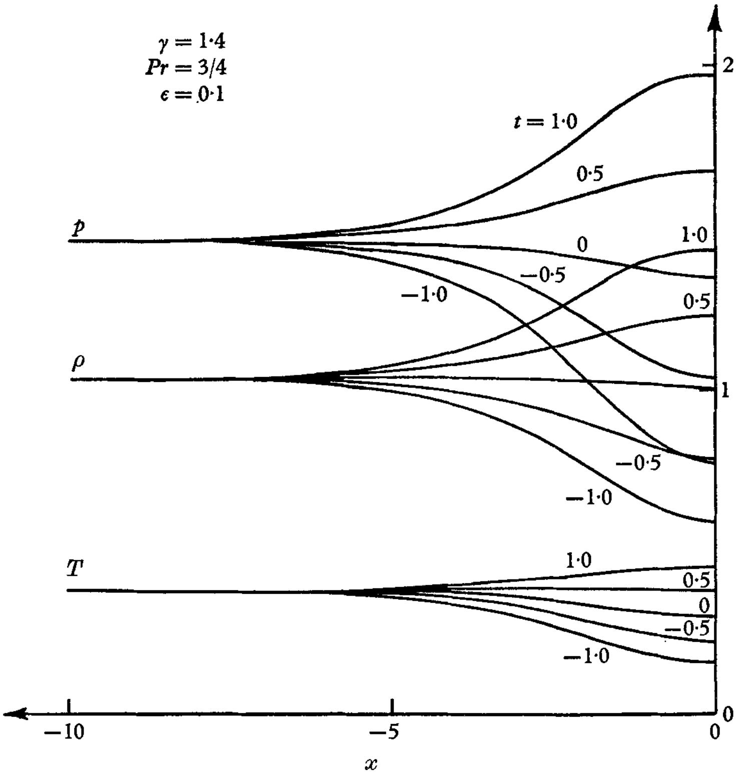

FIGURE 7. Pressure, density and temperature during reflexion as given by their composite expansions for an adiabatic wall.

Simplified model equations have proven useful in many areas of applied mathematics. We feel that the generalized Burgers equation may be of some general use because it gives a description of finite amplitude wave motions in which viscosity and convection in both wave directions play a role. Furthermore, we were able to find the co-ordinate stretchings that we employed directly from this model equation.

We conclude this paper by making note of some possible extensions of our work. One interesting application would be the study of shock formation by an accelerating piston. The work of Moran & Shen (1966), referred to in the introduction, considered the formation of a shock wave by a piston started impulsively. They showed that for small times the flow was governed by the linearized

† Agreement could be improved by modifying the definition of $\bar{t}$. Instead of letting $\bar{t} = \epsilon t$ we could use $(\bar{t})_{\mp} = \epsilon K_{\mp}(\epsilon) t$ where $(\bar{t})_-$ is used before reflexion and $(\bar{t})_+$ is used after reflexion. The terms K_+ and K_- are obtained from solving the conservation laws for the shock velocity as a function of ϵ, e.g. it is found that

$$K_-(\epsilon) = 1 + 2\{(1+\epsilon^2\Gamma^2/4)^{-\frac{1}{2}} - 1\}/\epsilon\Gamma = 1 + O(\epsilon).$$

Navier–Stokes equations, and that the solution of these equations could be matched with the solution of Burgers equation. They then found that the appropriate solution of Burgers equation was in fact uniformly valid for all time. The

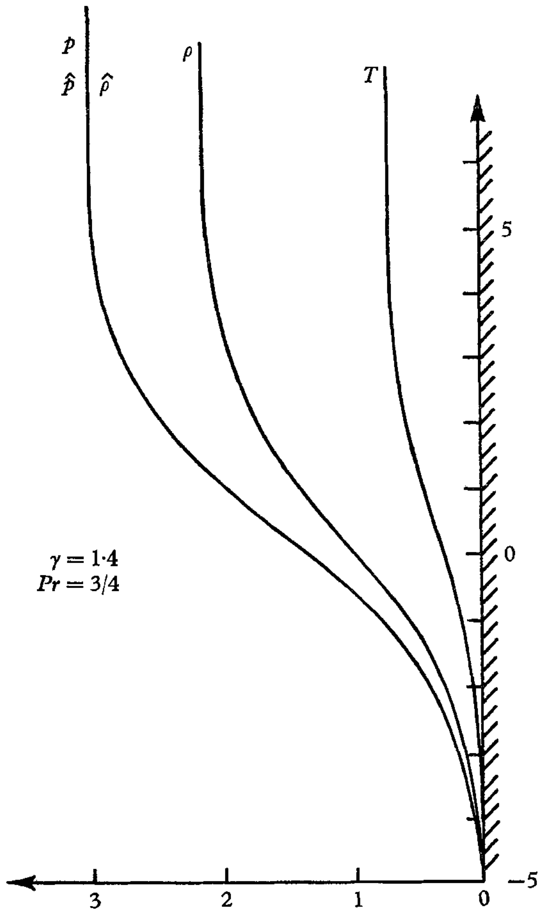

FIGURE 8. Pressure, density and temperature at the wall.

linear Navier–Stokes equations are considerably more difficult to deal with than is the inhomogeneous linear wave equation which results from our procedure; thus it would be quite difficult to deal with arbitrary piston accelerations using a linearized Navier–Stokes approach. On the other hand, the problem should be relatively easy to solve using our approach. The use of a thermal layer should permit one to handle a number of different heat transfer conditions at the piston surface. The problem of the structure of two unequal strength shocks undergoing a head-on collision should also be amenable to our methods. Finally, we expect that the method may be useful for other non-linear wave problems in which the wave is given structure by either diffusion or dispersion. An example of the latter type is provided by the non-linear dispersive wave equation studied by Zabusky (1967) in his work on wave motions in non-linear lattices:

$$y_{tt} - (1 + \epsilon y_x) y_{xx} - y_{xxxx} = 0. \qquad (6.1)$$

An analysis similar to that employed here for the reflecting shock problem shows that the problems encountered in a straightforward perturbation solution of this equation can be resolved by the method of matched asymptotic expansions. We find that a Burgers type region exists, and that the first term in an asymptotic

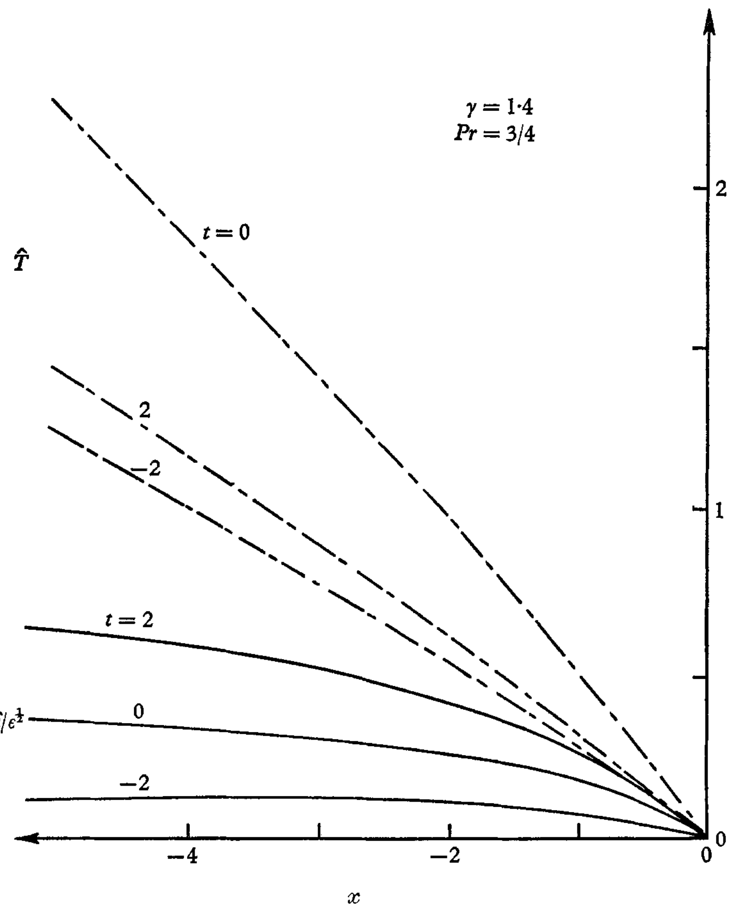

FIGURE 9. Temperature and velocity in the thermal layer.

expansion in this region is governed by the Korteweg–de Vries equation. Zabusky derived the Korteweg–de Vries equation from (6.1) by following the procedure used by Lighthill to derive Burgers equation. As the method seems to apply to Zabusky's equation as well as the the Navier–Stokes equations, we expect that it will apply to a number of finite amplitude wave problems where non-linear and dispersive or diffusive effects are important.

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Appendix

This appendix is devoted to showing that the first-order terms in the acoustic expansion of solutions of either the Navier–Stokes equation or the generalized Burgers equation are the same. We also show that second-order terms in the two expansions can be brought into agreement with a simple formula that corrects the temperature obtained from the generalized Burgers equation. Because the second-order velocity and density terms obtained from the generalized Burgers equation are shown to agree with the Navier–Stokes result, and the pressure is derived from the ideal gas law, a temperature correction is all that is needed to bring about complete agreement in the second-order acoustic expansions.

For the purposes of this appendix we denote the Navier–Stokes variables by the subscript $(\;)_{NS}$. The terms in the expansion of the solution to the generalized Burgers equation remain without subscripts. Because we assumed the isentropic gas law in the derivation of the generalized Burgers equation, this relationship is one of the system of equations used in §4. In expanded form this law is

$$\rho = (\gamma-1)^{-1}T + (2-\gamma)/[2(1-\gamma)^2]T^2 + O(\epsilon^2). \qquad \text{(A 1)}$$

The terms in the first-order acoustic expansion of a solution to these equations satisfy the system of equations

$$u_t^{(1)} + Q_x^{(1)} = 0, \quad Q_t^{(1)} + u_x^{(1)} = 0, \quad T^{(1)} = (\gamma-1)Q^{(1)}, \qquad \text{(A 2)}$$

and

$$\rho^{(1)} = T^{(1)}/(\gamma-1) = Q^{(1)}.$$

The first-order terms in the acoustic expansion of a solution of the Navier–Stokes equations satisfy

$$\rho^{(1)}_{NS_t} + u^{(1)}_{NS_x} = 0, \quad u^{(1)}_{NS_t} + (\rho^{(1)}_{NS_x} + T^{(1)}_{NS_x})/\gamma = 0, \qquad \text{(A 3)}$$

and

$$T^{(1)}_{NS_t} + (\gamma-1)u^{(1)}_{NS_x} = 0.$$

Eliminating $Q^{(1)}$ from (A 2) and differentiating the third equation, we find

$$\rho_t^{(1)} + u_x^{(1)} = 0, \quad T_t^{(1)} + (\gamma-1)u_x^{(1)} = 0. \qquad \text{(A 4)}$$

Because $(T^{(1)} + \rho^{(1)})/\gamma = T^{(1)}/(\gamma-1)$, we can write the first equation of (A 2) as

$$u_t^{(1)} + (T^{(1)} + \rho^{(1)})_x/\gamma = 0.$$

Equations (A 4) and (A 3) are identical. Because the boundary and initial conditions used for both sets of variables are the same, we have established equality between the first-order acoustic expansions obtained from the Navier–Stokes and the generalized Burgers equation provided we may assume the existence and uniqueness of the solution.

The second-order terms obtained from the generalized Burgers equation satisfy the equations

$$\left.\begin{aligned} u_t^{(2)} + Q_x^{(2)} &= 0,\\ Q_t^{(2)} + u_x^{(2)} &= \beta Q_{xx}^{(1)} - 2u^{(1)}Q_x^{(1)} - (\gamma-1)\,Q^{(1)}u_x^{(1)},\\ T^{(2)} &= (\gamma-1)[Q^{(2)} - u^{(1)^2}/2 + \beta u_x^{(1)}]\\ \text{and}\qquad \rho^{(2)} &= \frac{1}{\gamma-1}\left[T^{(2)} + \frac{2-\gamma}{2(\gamma-1)}T^{(1)^2}\right]. \end{aligned}\right\} \qquad \text{(A 5)}$$

From the Navier–Stokes equation we find that the second-order acoustic expansion satisfies the equations

$$\left.\begin{aligned}
\rho^{(2)}_{NS_t} + u^{(2)}_{NS_x} &= -(\rho^{(1)}_{NS} u^{(1)}_{NS})_x \equiv F_{NS_1},\\
u^{(2)}_{NS_t} + (\rho^{(2)}_{NS} + T^{(2)}_{NS})_x/\gamma &= u^{(1)}_{NS_{xx}} - \rho^{(1)}_{NS} u^{(1)}_{NS_t}\\
&\quad - u^{(1)}_{NS} u^{(1)}_{NS_x} - (\rho^{(1)}_{NS} T^{(1)}_{NS})_x/\gamma \equiv F_{NS_2},\\
T^{(2)}_{NS_t} + (\gamma-1) u^{(2)}_{NS_x} &= \gamma T^{(1)}_{NS_{xx}}/Pr - \rho^{(1)}_{NS} T^{(1)}_{NS_t}\\
&\quad - u^{(1)}_{NS} T^{(1)}_{NS_x} - (\gamma-1)(\rho^{(1)}_{NS} + T^{(1)}_{NS}) u^{(1)}_{NS_x} \equiv F_{NS_3}.
\end{aligned}\right\} \quad (A\ 6)$$

Equations (A 5) can be put into the form

$$\left.\begin{aligned}
\rho^{(2)}_t + u^{(2)}_x &= -(\rho^{(1)} u^{(1)})_x \equiv F_1,\\
u^{(2)}_t + \frac{1}{\gamma}(T^{(2)} + \rho^{(2)})_x &= \beta u^{(1)}_{xx} - u^{(1)} u^{(1)}_x + \frac{2-\gamma}{\gamma}\rho^{(1)}\rho^{(1)}_x \equiv F_2\\
\text{and} \quad T^{(2)}_t + (\gamma-1) u^{(2)}_x &= -(\gamma-1)^2 \rho^{(1)} u^{(1)}_x - (\gamma-1) u^{(1)} \rho^{(1)}_x \equiv F_3.
\end{aligned}\right\} \quad (A\ 7)$$

If the second-order generalized Burgers and Navier–Stokes variables are equated, the first of equations (A 6) and (A 7) agree, but the second and third of equations (A 6) do not correspond with (A 7). Nevertheless, by eliminating variables, we can derive the following equations for $u^{(2)}_{NS}$ and $u^{(2)}$:

$$u^{(2)}_{NS_{tt}} - u^{(2)}_{NS_{xx}} = (F_{NS_2})_t - \frac{1}{\gamma}(F_{NS_1} + F_{NS_3})_x, \quad (A\ 8)$$

$$u^{(2)}_{tt} - u^{(2)}_{xx} = F_{2_t} - \frac{1}{\gamma}(F_1 + F_3)_x. \quad (A\ 9)$$

Since the first-order solutions are the same, we know that the right sides of (A 8) and (A 9) are equal. Because $u^{(2)}$ and $u^{(2)}_{NS}$ satisfy identical boundary conditions and equations, they must be equal. The first of equations (A 6) and (A 7) shows that $\rho^{(1)}$ corresponds to $\rho^{(1)}_{NS}$. By eliminating $u^{(2)}_{NS}$ from the third equation of (A 6), we find the equation,

$$(T^{(2)}_{NS})_t = (\gamma-1)\rho^{(2)}_{NS} - (\gamma-1)F_{NS_1} + F_{NS_3}.$$

With $$F_{NS_1} = F_1 \quad \text{and} \quad F_{NS_3} = F_3 + (\beta-1)\gamma\rho^{(1)}_{xx},$$

we finally arrive at the correction formula for $T^{(2)}$:

$$(T^{(2)}_{NS} - T^{(2)})_t = \frac{\gamma}{Pr} T^{(1)}_{xx}. \quad (A\ 10)$$

If there is no heat conduction, $Pr \to \infty$ and $T^{(2)}_{NS} \to T^{(2)}$. This last result could have been predicted on physical grounds because the viscous term in the Navier–Stokes equation does not contribute to entropy transport, only to entropy production. This last effect is third order, and could have no effect on the second-order acoustic expansion.

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